value of the rate of propagation of the long waves at the boundary of separation corresponds to it.

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ON VORTICITY-INDUCED WAVES IN A HOMOGENEOUS INCOMPRESSIBLE FLUID*

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The existence of vortex-induced waves in a homogeneous incompressible fluid is proved. The boundary of the vortex represents a cylindrical rotating fluid surface of stable form. The non-linear dispersion relation, the form of the vortex and the stream function are found for the vortices bounded by an almost circular cylinder, and for the vortex-induced waves. The character and special features of the oscillation of the velocity field are explained.

The problem is reduced to that of proving the existence of a branching solution of the non-linear integral equation and to effective determination of the solution and the bifurcation value of the parameter. An iterative method is proposed enabling the simultaneous determination at every stage of the approximation to the branching solution and bifurcation value of the parameter. The convergence of the method over a certain range of parameters is proved.

The possibility of the existence of rotating cylindrical vortices retaining the non-circular form of the transverse cross-section was shown by Lamb /1/ who obtained the linearized dispersion relation (3.2). Following /2/ we shall call such vortices "vortons". Deem and Zabusky carried out a numerical experiment in /2/ and they suggest that the result proves the existence of vortons. It was also found that the rotation frequency of these vortices is less than the value obtained from (3.2).

The vortons and vorton-induced waves are of interest (see the foreword to /2/), since the results of the numerical experiment are interpreted as manifestations of the "soliton-like" behaviour of the waves in a two-dimensional medium.

1. Formulation of the problem. Consider the flow of an ideal homogeneous incompressible and unbounded fluid in a direction parallel to the xoy plane (Fig.1). We denote by ox, oy the fixed axes and by ox_1, oy_1 the axes rotating with constant angular velocity Ω, r. θ are polar coordinates in the xoy plane, r, β are polar coordinates in the x_1oy_1 plane, t is time, $\mathbf{q}(r, \theta, t)$ denotes the absolute velocity of the fluid (relative to the fixed axes), $q_r, q_\theta \equiv q_\beta$ is the radial and tangential component of the absolute velocity, and $\zeta = \mathrm{rot}\,\mathbf{q} = \zeta \mathbf{i}_z$, $\mathbf{i}_z$ is the unit vector normal to the xoy plane (and to the x_1oy_1) plane). When $t = 0$, the ox and ox_1 axes coincide.

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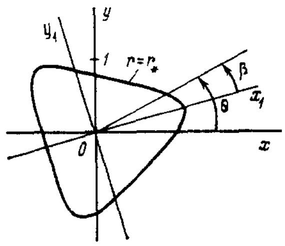

Fig.1

Suppose $\zeta = 1$ within the fluid volume bounded by the cylinder $r = R_*(\theta, t)$ ($r < R_*$ inside the cylinder) and $\zeta = 0$ outside it. We shall call a fluid volume with non-zero vorticity, a vortex.

Under these assumptions a stream function Ψ exists connected with the vorticity $\zeta = 1$ by the relation

$$\Psi = \frac{1}{4\pi}\iint \ln[r^2 + r_1^2 - 2rr_1\cos(\theta_1 - \theta)]\, r_1\, dr_1\, d\theta_1 \tag{1.1}$$

The integration is carried out over the transverse cross-section of the vortex. We shall investigate a vortex whose boundary rotates with constant angular velocity Ω retaining its form, so that $R_*(\theta, t) = r_*(\beta)$. We use the formulas $r = \rho r_*(\beta)$, $\beta = \theta - \Omega t$ to replace the variables r and θ by ρ and β respectively. In the new variables the vortex boundary is described by the equation $\rho = 1$, and its interior by the inequality $0 \leqslant \rho < 1$, $0 \leqslant \beta < 2\pi$.

Relation (1.1) now becomes

$$\Psi(\rho, \beta) = \frac{1}{4\pi}\iint_{\rho_1<1} K(\rho, \rho_1, \beta, \beta_1)\, \rho_1 r_*^2(\beta_1)\, d\rho_1\, d\beta_1 \tag{1.2}$$

$$K = \ln W$$

$$W = \rho^2 r_*^2(\beta) + \rho_1^2 r_*^2(\beta_1) - 2\rho\rho_1 r_*(\beta)\, r_*(\beta_1)\cos(\beta_1 - \beta)$$

The function $\Psi(\rho, \beta)$ determines the flux across the cylindrical surface fixed with respect to the absolute xoy axes. We introduce the stream function F which determines the flux across the cylindrical surface fixed with respect to the rotating x_1oy_1 axes. If $P(r, \theta)$ is a point in the xoy plane (i.e. P is fixed relative to the xoy axes) and $Q(r, \beta)$ is a point in the x_1oy_1 plane coinciding at the given instant t with P (i.e. $\beta = \theta - \Omega t$), then the two stream functions at the corresponding points are connected by the relation

$$F(Q) = \Psi(P) - {}^1/_2\Omega r^2$$

The vortex boundary is fixed with respect to the x_1oy_1 axes and the vorticity of the fluid particle is preserved. This implies that $F = \text{const}$ at the vortex boundary, i.e.

$$\Psi(1;\ \beta) - {}^1/_2\,\Omega r_*^2(\beta) = c_0 = \text{const} \tag{1.3}$$

Relation (1.3) represents a non-linear integral equation for the function $r_*(\beta)$. An integro-differential equation for $r_*(\beta)$ was constructed and used in /2/. It can be confirmed that after integrating along the boundary, the equation leads to (1.3).

The equations (1.2), (1.3) are written in dimensionless coordinates. The dimensional unit length R_0 and the time T_0 are chosen from the conditions

$$R_0^2\int_0^{2\pi} r_*^2(\beta)\, d\beta = 2S, \quad \zeta = 1 \tag{1.4}$$

Here S is the area of transverse cross-section of the cylindrical vortex and $\zeta_* = T_0^{-1}\zeta$ represents the dimensional vorticity. If the vortex is bounded by a circular cylinder, then R_0 is the radius of the cylinder and $r_* = 1$.

Equation (1.3) has a solution $r_* = 1$ for any Ω. Let us inspect other (non-trivial) solutions where $r_* \neq \text{const}$. The solutions correspond to the bifurcation values of the parameter Ω.

2. On the integral operator and certain integrals. We use the properties of the following linear integral operator * in constructing a non-trivial solution of the problem

$$T(f(\beta)) = \frac{1}{4\pi}\int_0^{2\pi} \ln[\rho^2 + \rho_1^2 - 2\rho\rho_1\cos(\beta_1 - \beta)]\, f(\beta_1)\, d\beta_1 \tag{2.1}$$

Its eigenvalues $\lambda_k(\rho, \rho_1)$ are given by the formulas

$$\rho_1 < \rho, \quad \lambda_0 = \ln\rho, \quad \lambda_k = -\frac{1}{2k}\frac{\rho_1^k}{\rho^k} \tag{2.2}$$

$$\rho_1 > \rho, \quad \lambda_0 = \ln\rho_1, \quad \lambda_k = -\frac{1}{2k}\frac{\rho^k}{\rho_1^k} \quad (k = 1, 2, \ldots)$$

The eigenvalue $\lambda_k(\rho, \rho_1)$ has the corresponding pair of eigenfunctions $\cos k\beta$ and $\sin k\beta$, and hence their linear combination. The following relation holds:

* Mindlin I.M. On the vortices in an unbounded ideal fluid. Gor'kii, Dep. at VINITI, 24.6.82, No.3269-82, 1982.

$$T\left(c_1 \cos k\beta+c_2 \sin k\beta\right)=\lambda_k\left(c_1 \cos k\beta+c_2 \sin k\beta\right) \tag{2.3}$$

The eigenfunction $f_0 = 1$ corresponds to the eigenvalue λ_0.
Below we shall use the operator

$$T_1(f(\beta))=T(f(\beta))|_{\rho=\rho_1=1}=\frac{1}{4\pi} \int_0^{2\pi} \ln\left[2-2 \cos \left(\beta_1-\beta\right)\right] f\left(\beta_1\right) d\beta_1 \tag{2.4}$$

and the formulas for the integrals

$$T_{kn}=\int_0^{2\pi} \frac{\cos k\alpha}{\left(\rho^2+\rho_1{}^2-2\rho\rho_1 \cos \alpha\right)^n} d\alpha \quad(k=0,1,2, \ldots)$$

In particular, we have

$$\rho_1<\rho, \quad T_{k1}=F_1\left(\rho, \rho_1\right)=\frac{2\pi}{\rho^2-\rho_1{}^2} \frac{\rho_1{}^k}{\rho^k} \tag{2.5}$$

$$T_{k2}=F_2\left(\rho, \rho_1\right)=\frac{2\pi}{\left(\rho^2-\rho_1{}^2\right)^2}\left[k+\frac{\rho^2+\rho_1{}^2}{\rho^2-\rho_1{}^2}\right] \frac{\rho_1{}^k}{\rho^k}$$

$$T_{k3}=F_3\left(\rho, \rho_1\right)=$$

$$\frac{\pi}{\left(\rho^2-\rho_1{}^2\right)^3}\left[k^2+3k \frac{\rho^2+\rho_1{}^2}{\rho^2-\rho_1{}^2}+2 \frac{\rho^4+4\rho^2\rho_1{}^2+\rho_1{}^4}{\left(\rho^2-\rho_1{}^2\right)^2}\right] \frac{\rho_1{}^k}{\rho^k}$$

$$\rho_1>\rho, \quad T_{k1}=F_1\left(\rho_1, \rho\right), \quad T_{k2}=F_2\left(\rho_1, \rho\right), \quad T_{k3}=F_3\left(\rho_1, \rho\right)$$

The derivation of (2.5) is lengthy and is therefore omitted. Formulas (2.2) are obtained by integrating $T(\cos k\beta)$ by parts and applying (2.5).

3. Existence of a solution.

The question of the existence of a solution of (1.2), (1.3) is of fundamental importance, because heuristically it is not at all clear whether a non-circular cylindrical liquid surface rotating, as if it was a rigid surface, exists.

Th proof of the existence of a branching solution of the integral equation is usually carried out in two stages /3/: a) a free parameter is introduced into the equation and an iterative method using the theorem of the fixed point of compressive mapping is employed to show that a family of solutions depending on the parameter exits; b) the existence of a solution of the branching equation containing the parameter as the unknown is proved. To construct the branching equation we must in fact first obtain the family of solutions of the integral equation. The procedure can also be applied to the problem (1.2), (1.3).

Below we present an iterative method in which we obtain, at every step, the approximation to the branching solution as well as the approximation to the bifurcation value of the parameter. The advantages of this method in solving the problem numerically are obvious. The properties of the operator (2.1) and formulas (2.5) are used to show the convergence of the iterations within a certain range of parameter values.

Let us describe the method. We use the relations

$$r_*(\beta)=1+\varepsilon \cos m\beta+\varepsilon^2 u(\beta), \quad \Omega=\Omega_0+\varepsilon^2 \omega \tag{3.1}$$

to introduce the parameter ε into (1.3), the function $u(\beta)$ to replace $r_*(\beta)$ and the parameter ω to replace Ω; $m>1$ is an integer. After the substitution (3.1) we write the integrand in the form $h_0+\varepsilon h_1+\varepsilon^2 h_2+\varepsilon^3 h_3$ where h_0, h_1, h_2 does not contain ε explicitly (the solution $u(\beta)$ depends on ε implicitly). Let us write

$$\Omega_0=\frac{1}{2}\left(1-\frac{1}{m}\right), \quad c_0=-\frac{1}{2} \Omega_0 \tag{3.2}$$

When the constants are chosen in this manner, the integral equation will not contain ε explicitly in the zero and first power. Dividing by ε^2 and evaluating the integrals that are independent of $u(\beta)$), we write the integral equation in the form

$$\frac{1}{2m} u(\beta)+T_1(u(\beta))=H_0(\omega)+\varepsilon H_1(u(\beta), \omega, \varepsilon) \tag{3.3}$$

$$H_0(\omega)=-\frac{1}{8m}+\left(\frac{1}{8}-\frac{1}{16m}\right) \cos 2m\beta+\frac{1}{2} \omega$$

Without writing out a lengthy formula for H_1, we note that

$$H_1(u(\beta), \omega, 0)=\left[\Omega_0 u(\beta)+\omega+{}^1/{}_8\, m-\frac{3}{16}\right] \cos m\beta- \tag{3.4}$$

$$\frac{1}{48} \cos 3m\beta-2 \int_0^1 \rho_1 T(u(\beta) \cos m\beta)|_{\rho=1} d\rho_1+S(u(\beta))$$

Here $S(u)$ is a linear integral equation (3.3) with the following properties:

$$k \geqslant 1, \quad S(\cos km\beta) = d_{k-1}\cos(k-1)m\beta + d_{k+1}\cos(k+1)m\beta \tag{3.5}$$

$$d_k = \frac{1}{4}\frac{11}{mk+2}, \quad d_0 = -\frac{1}{4} + \frac{1}{m+2}$$

$$S(1) = -\frac{1}{2}\left(1 - \frac{1}{m+2}\right)\cos m\beta, \quad S(0) = 0$$

Using (2.1)—(2.5) we find, that equation (3.3) with $\varepsilon = 0$ has the following solution (with an arbitrary ω_0):

$$u(\beta) = u_0(\beta) = g_0 + g_2\cos 2m\beta \tag{3.6}$$

$$g_0 = -\frac{1}{4} + m\omega_0, \quad g_2 = \frac{1}{2}m - \frac{1}{4}$$

The solution $u(\beta, \varepsilon)$ which tends to $u_0(\beta)$ and $\varepsilon \to 0$, corresponds to the non-trivial solution of the problem (1.2), (1.3). We shall obtain this solution (beginning from u_0(3.6)) by interation

$$\frac{1}{2m}u_{k+1}(\beta) + T_1(u_{k+1}(\beta)) = H_0(\omega_k) + \varepsilon H_1(u_k(\beta), \omega_k, \varepsilon) \tag{3.7}$$

$$\int_0^{2\pi}[H_0(\omega_k) + \varepsilon H_1(u_k(\beta), \omega_k, \varepsilon)]\cos m\beta\, d\beta = 0 \tag{3.8}$$

According to condition (3.8), the value ω_k is selected so that the expansion of the right-hand side of the equation (3.7) and the corresponding Fourier series do not contain the harmonic $\cos m\beta$, otherwise (3.7) will have no solution.

The sequences u_k, ω_k converge to the solution of (3.3).

To show the convergence of the sequences u_k, ω_k (the unboundedness of the kernel of the operator H_1 complicates the problem), we shall first consider the iterative process linearized in ε

$$\frac{1}{2m}u_{k+1}(\beta) + T_1(u_{k+1}(\beta)) = H_0(\omega_k) + \varepsilon H_1(u_k(\beta), \omega_k, 0) \tag{3.9}$$

$$\int_0^{2\pi}[H_0(\omega_k) + \varepsilon H_1(u_k(\beta), \omega_k, 0)]\cos m\beta d\beta = 0$$

From (2.1)—(2.5), (3.4)—(3.6) it follows that the process (3.9) leads to approximations of the form

$$u_k(\beta) = g_0 + g_2\cos 2m\beta + \sum_{j=0}^{k+2} b_j^{(k)}\cos jm\beta, \quad b_1^{(k)} = 0 \tag{3.10}$$

and we have

$$\omega_k - \omega_0 = -{}^1/_4 b_2^{(k)}, \; b_0^{(k+1)} = m(\omega_k - \omega_0) \tag{3.11}$$

$$b_2^{(k+1)} = m\varepsilon(1 - 1/2m)\, b_2^{(k)}$$

$$b_3^{(k+1)} = m\varepsilon\left[\left(\frac{3}{4} - \frac{1}{2m}\right)(g_2 + b_2^{(k)} + b_4^{(k)}) - \frac{1}{16}\right]$$

$$j > 3, \quad b_j^{(k+1)} = m\varepsilon\frac{j}{j-1}\left(\frac{1}{mj} + 2\Omega_0\right)\cdot\frac{1}{2}(b_{j+1}^{(k)} + b_{j-1}^{(k)})$$

$$j > k+2, \; b_j^{(k)} = 0; \; j \geqslant 0, \; b_j^{(0)} = 0$$

From (3.11) it follows that $b_j^{(k)} \geqslant 0$ when $j \geqslant 2\,(\varepsilon > 0)$

$$|b_j^{(k+1)} - b_j^{(k)}| < \frac{3}{8}m\left(\frac{3}{2}m\varepsilon\right)^k$$

$$\sum_{j=2}^{k+3}|b_j^{(k+1)} - b_j^{(k)}| < \frac{3}{8}m(k+2)\left(\frac{3}{2}m\varepsilon\right)^k$$

From the estimates obtained we conclude that the process (3.9) determines the sequences ω_k and $u_k(\beta)$ converging for small $m\varepsilon$. The operator $H_1(u, \omega, \varepsilon)$ differs from the operator $H_1(u, \omega, 0)$ by a term of order ε.

Estimates analogous to those obtained above show that additional of higher-order infinitesimals does not violate (for small ε) the convergence of the iterative process, since the "unperturbed" process (3.9) converges at least as fast as a geometrical progression.

The proof of the fact that the limit of the iterations (3.7), (3.8) is a solution of (3.3), is a repeat of known cases, and is not given here. We note, that what was said above, can be reformulated in terms of the transformations, and the proof given establishes that for small $m\varepsilon$ formulas (3.7), (3.8) determine the compression mapping.

4. Non-linear dispersion relation. The solution of (1.2), (1.3) can be constructed in terms of series in integer, non-negative powers of ε, with conditions of the type (3.8). The first three terms of these series agree with the results of the first iteration (3.6)–(3.8). The first iteration yields the formulas

$$\Omega = \Omega_0 + \varepsilon^2\omega_0 + o(\varepsilon^3), \quad \omega_0 = -\frac{1}{4}(m-1) \tag{4.1}$$

$$r_*(\beta) = 1 + \varepsilon\cos m\beta + \varepsilon^2(g_0 + g_2\cos 2m\beta) + \varepsilon^2 g_3\cos 3m\beta + o(\varepsilon^3) \tag{4.2}$$

$$g_0 = -\frac{1}{4}(m^2 - m + 1), \quad g_2 = \frac{1}{2}m - \frac{1}{4},$$

$$g_3 = \frac{1}{8}(m = 1)(3m - 1)$$

$$\Psi(\rho, \beta) = \Psi_0 + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \varepsilon^3\Psi_3 + o(\varepsilon^3) \tag{4.3}$$

$$\rho < 1, \quad \Psi_0 = -\frac{1}{4}(1 - \rho^2); \quad \rho > 1, \quad \Psi_0 = \frac{1}{2}\ln\rho$$

$$\Psi_1 = \Psi_{1,1}\cos m\beta, \quad \Psi_2 = \Psi_{2,0} + \Psi_{2,2}\cos 2m\beta$$

$$\rho < 1, \quad \Psi_{1,1} = \frac{1}{2}\rho^2 - \frac{1}{2m}\rho^m$$

$$\Psi_{2,2} = \frac{1}{4}m\rho^2 - \frac{1}{4}\rho^m$$

$$\Psi_{2,0} = \left(\frac{1}{2}g_0 + \frac{1}{8}\right)\rho^2 + \frac{1}{4} - \frac{1}{4}\rho^m$$

$$\rho > 1, \quad \Psi_{1,1} = \frac{1}{2} - \frac{1}{2m}\rho^{-m}$$

$$\Psi_{2,2} = \frac{1}{4}(m-1) + \frac{1}{4}\rho^{-m} - \frac{1}{4}\rho^{-2m}$$

$$\Psi_{2,0} = \frac{1}{2}g_0 - \frac{1}{8} + \left(g_0 + \frac{1}{4}\right)\ln\rho + \frac{1}{4}\rho^{-m}$$

$$\Psi_3 = \Psi_{3,1}\cos m\beta + \Psi_{3,3}\cos 3m\beta$$

$$\rho < 1, \quad \Psi_{3,1} = \left(-\frac{1}{8}m^2 + \frac{1}{4}m - \frac{3}{16}\right)\rho^2 + \left(-\frac{1}{8}m + \frac{3}{16m}\right)\rho^m$$

$$\Psi_{3,3} = \frac{1}{2}g_3\left(\rho^2 - \frac{1}{3m}\rho^{3m}\right) + \frac{1}{4}g_2\left[\rho^2 - \rho^m - \rho^{2m} + \left(1 - \frac{1}{3m}\right)\rho^{3m}\right] - \frac{1}{16}(m-1)\rho^m + \frac{1}{16}(2m-1)\rho^{2m} - \frac{1}{48}(3m-1)\rho^{3m}$$

$$\rho > 1, \quad \Psi_{3,1} = -\frac{1}{8}m^2 + \frac{5}{16} + \frac{1}{4}m\rho^{-2m} + \left(-\frac{1}{8}m - \frac{1}{2} + \frac{3}{16m}\right)\rho^{-m}$$

$$\Psi_{3,3} = \frac{1}{2}g_3\left(1 - \frac{1}{3m}\rho^{-3m}\right) + \frac{1}{4}g_2\left[-1 + \rho^{-m} + \rho^{-2m} - \left(1 + \frac{1}{3m}\right)\rho^{-3m}\right] + \frac{1}{24} - \frac{1}{16}(m+1)\rho^{-m} + \frac{1}{16}(2m+1)\rho^{-2m} - \frac{1}{48}(3m+1)\rho^{-3m}$$

To obtain the representation indicating the limits of applicability of the formulas obtained, we will compare the numerical and analytic results. The following values for the vortex boundary parameters were found in /2/ for the given values of the vortex area $S = 5$, its period of rotation $T = 20$ and wave number $m = 3$: the smallest and largest radius $R_1 = 1.056$ and $R_2 = 1.591$, and the amplitudes of the harmonics $a_0 = 1.250$, $a_m = 2.394\cdot 10^{-1}$, $a_{2m} = 6.08\cdot 10^{-2}$, $a_{3m} = 2.04\cdot 10^{-2}$.

Let us find the corresponding parameters for the analytic solution. When $m = 3$, (4.2) yields

$$r_*(\beta) = 1 + \varepsilon\cos 3\beta + \varepsilon^2(-1.75 + 1.25\cos 6\beta) + 2\varepsilon^3\cos 9\beta \tag{4.4}$$

When $|\varepsilon| < 0.4$, the right-hand side of (4.4) attains minimum r_1 at $\beta = \pi/3$ and maximum r_2 at $\beta = 0$, so that $r_2 - r_1 = 2\varepsilon + 4\varepsilon^3$. We find R_0 and ε from the conditions $R_0(r_2 - r_1) = R_2 - R_1 = 0.535$, $\pi R_0^2(1 - 3\varepsilon^2) = 5$, where the latter is obtained form (1.4), (4.4). Thus for the numerical solution we have the corresponding analytical solution with parameters $R_0 = 1.334$ and $\varepsilon = 0.187$. According to (4.1) we have $T = 2\pi/\Omega = 19.90$ for the analytical solution. Neglecting the correction term $\varepsilon^2\omega_0$ we obtain $T = 18.86$, and the following values for the amplitudes of the harmonics: $a_0 = (1 - 1.75\ \varepsilon^2)\ R_0 = 1.252$, $a_m = R_0\varepsilon = 2.499\cdot 10^{-1}$, $a_{2m} = 1.25\ R_0\varepsilon^2 = 5.855\cdot 10^{-2}$, $a_{3m} = 2\varepsilon^3 R_0 = 1.756\cdot 10^{-2}$.

Another three sets of parameters were obtained by numerical methods for $m = 3$, $T = 20.5$; $m = 4$, $T = 17.0$; $m = 4$, $T = 17.5$ (in all cases we had $S = 5$). The corresponding parameters of the analytic solution were $\varepsilon = 0.2184$, $T = 20.30$; $\varepsilon = 0.0925$, $T = 17.04$; $\varepsilon = 0.1399$, $T = 17.44$.

Thus the non-linear dispersion relation exhibits a high degree of accuracy within the range $|m\varepsilon| \leqslant 0.5$. Formula (4.2) yields, in this range, result differing from the numerical results by 5—7%. Fig.1 depicts the vortex boundary constructed from (4.4) for $\varepsilon = 0.2$.

5. Velocity field. The character and specific features of the oscillations.

The components of the absolute velocity are connected with the stream function by the relations

$$q_r = -\frac{1}{\rho r_*}\frac{\partial\Psi}{\partial\beta} + \frac{1}{r_*^2}\frac{dr_*}{d\beta}\frac{\partial\Psi}{\partial\rho}, \quad q_\beta = \frac{1}{r_*}\frac{\partial\Psi}{\partial\rho} \tag{5.1}$$

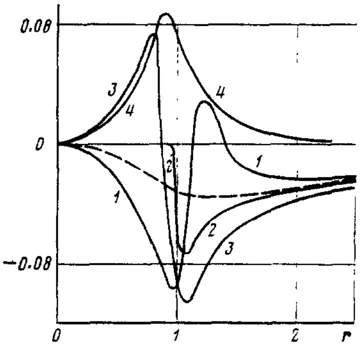

Fig.2

Fig.2 shows curves constructed from formulas (4.3), (5.1) for $m = 3$, $\varepsilon = 0.2$. The curves show the distribution of the quantities q_r and $Q_\beta = q_\beta - q_0(r)$ as a function of the radius r, for fixed θ and over a quarter period time interval, for the phases $\beta = 0, \beta = \pi/6, \beta = \pi/3, \beta = \pi/2$ where $q_0(r) = \partial\Psi_0(r)/\partial r$ is the tangential component of the velocity of the fluid particles of the circular vortex (i.e. in the case when $\varepsilon = 0$). Curves *1—3* depict $Q_\beta(r)$ at $\beta = 0$, $\beta = \pi/6$, $\beta = \pi/3$ respectively. When $\beta = \pi/2$, the graph of $Q_\beta(r)$ coincides with curve *2*. Curve *4* depicts $q_r(r)$ for $\beta = \pi/2$; $q_r = 0$ with $\beta = 0$ and $\beta = \pi/3$. When $\beta = \pi/6$, the graph of the function $q_r(r)$ and curve *4* are mutually symmetrical with respect to the r axis. The curves also represent the graphs of $Q_\beta(r)$ and $q_r(r)$ at the same instant t, with an interval of the angle θ equal to $\Delta\theta = \pi/6$. The interval $\Delta\theta$ represents a quarter of the angular period of the oscillating velocity field. The dashed line shows the time-averaged position of the curves $Q_\beta(r)$. The graphs show that the vortex induced waves are localized near the vortex within 2—3 of the mean radii R_0.

It can be shown that not only the approximate solution, but also the exact solution of the problem is invariant with respect to rotation by an angle $\theta = 2\pi/m$.

The results obtained show that the system in question is different from the many known conservative mechanical oscillating systems. In a typical mechanical system undergoing oscillation, the kinetic energy is converted over a half-period into the potential energy of an "elastic element" and vice versa. The graphs show that in the system in question the maximum deviations of the velocity components from their mean value are reached with a corresponding shift in time and space of approximately a quarter of a period, so that the kinetic energy is "pumped" during the oscillations from radial to tangential displacements and vice versa.

Finally we note that the operator (2.1) and formulas (2.5) can be used to construct a class of steady cylindrical vortices in which the vorticity and stream function are connected by a non-linear relation, and a class of non-steady vortices in a heavy fluid with piecewise-constant density, just as was done * (*see the previous footnote.) for axisymmetric vortices.

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